ENGINEERING EXPERIMENT STATION
of the Georgia Institute of Technology
Atlanta, Georgia

FINAL TECHNICAL REPORT

PROJECT NO. A-241-4

OPTIMUM USE OF LIMITED MICROWAVE APERTURES

by

ROGER D. WETHERINGTON

DJ

- o - o - o - o -

DEPARTMENT OF THE NAVY
OFFICE OF NAVAL RESEARCH
CONTRACT NO. NOnr-991(02)
SUB-TASK NO. 4

- o - o - o - o -

29 JUNE 1956

ENGINEERING EXPERIMENT STATION
of the Georgia Institute of Technology
Atlanta, Georgia

FINAL TECHNICAL REPORT

PROJECT NO. A-241-4

OPTIMUM USE OF LIMITED MICROWAVE APERTURES

by

ROGER D. WETHERINGTON

- o - o - o - o -

DEPARTMENT OF THE NAVY
OFFICE OF NAVAL RESEARCH
CONTRACT NO. NOnr-991(02)
SUB-TASK NO. 4

- o - o - o - o -

29 JUNE 1956

TABLE OF CONTENTS

This Report Contains 21 Pages

I. INTRODUCTION

The purpose of this study was to investigate the possibility of improving the angular resolution of tracking or search radars by operating on the incoming signals in a manner different from that ordinarily used. The basic ideas cited here first arose in a mathematical analysis of the signals present in the plane of an aperture which was carried out under contract NOrd 11224.

A. Simultaneous Lobe Comparison.

Consider a simultaneous lobe comparison (SLC) tracking radar in two dimensions. Let AB (Figure 1) represent one of the two apertures of the system,

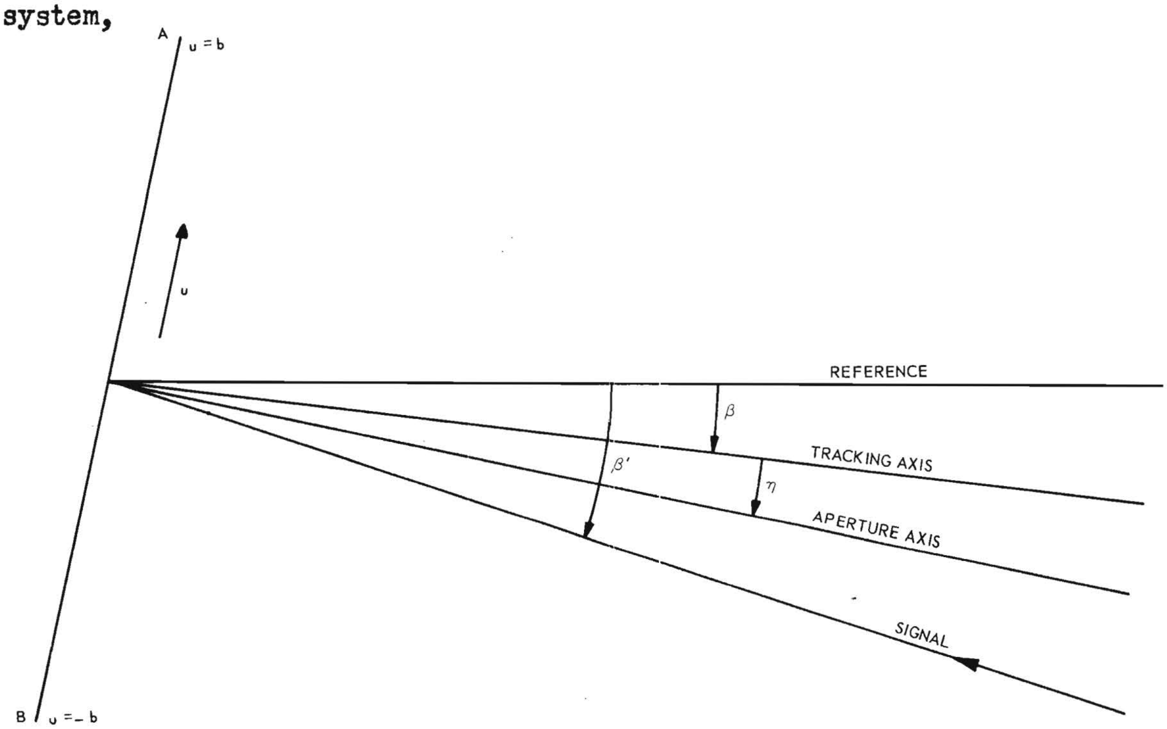

Figure 1. Antenna Geometry

and let

β = angle from the reference line to the tracking axis,
$\eta = \frac{1}{2}$ the squint angle,
β_1 = angle of arrival of an incoming signal,

u = distance from the center to any point on the aperture, and

b = $\frac{1}{2}$ the aperture width.

For a signal of amplitude a, the field at any point in the aperture will be

(1) $$S(u,\beta,\beta_1,t) = a \cos [\omega_1 t + ku \sin (\beta + \eta - \beta_1) + \xi_1]$$

where

ω_1 = frequency of the incoming wave,

ξ_1 = phase angle of the incoming wave, and

$k = \frac{2\pi}{\lambda}$.

Since $(\beta + \eta - \beta_1)$ will usually be quite small, assume that

$$\sin (\beta + \eta - \beta_1) = (\beta + \eta - \beta_1) \quad ,$$

and

(2) $$S(u,\beta,\beta_1,t) = a \cos [\omega_1 t + ku(\beta + \eta - \beta_1) + \xi_1] \quad .$$

The aperture output will be

$$S_+(\beta,\beta_1) = \int_{-b}^{b} S(u,\beta,\beta_1,t)\, du$$

(3) $$= 2b \frac{\sin k(\beta + \eta - \beta_1)b}{k(\beta + \eta - \beta_1)b} \cos (\omega_1 t + \xi_1) \quad .$$

The output of the other aperture will be identical except that η will be replaced by $-\eta$, therefore

(4) $$S_-(\beta,\beta_1) = 2b \frac{\sin k(\beta - \eta - \beta_1)b}{k(\beta - \eta - \beta_1)b} \cos (\omega_1 t + \xi_1) \quad .$$

Neglectiong AGC, the error signal will be the time average of $(S_+^2 - S_-^2)$ or

(5) $$F(\beta,\beta_1) = 2b^2 \left[\frac{\sin^2 k(\beta + \eta - \beta_1)b}{[k(\beta + \eta - \beta_1)b]^2} - \frac{\sin^2 k(\beta - \eta - \beta_1)b}{[k(\beta - \eta - \beta_1)b]^2}\right] .$$

B. Dot Product of the Signal in the Aperture

Let us assume that we have access to the signal at each point in the aperture. Let the signal at each point be separated into two parts of equal amplitude. By introducing a 90° phase shift and then attenuating one-half of the signal we obtain the two functions

(6) $$\sigma(u) = \frac{a}{2} \cos [\omega t + k(\beta + \eta - \beta_1) + \xi_1] ,$$

(7) $$\delta(u) = \frac{u}{b} \left(\frac{a}{2}\right) \sin [\omega t + k(\beta + \eta - \beta_1) + \xi_1] .$$

Now consider a function defined by

(8) $$\tilde{F}(\beta,\beta_1) = \int_{-b}^{b} \sigma(u)\ \delta(-u)\ du$$

(9) $$= \frac{1}{2k} \left(\frac{a}{2}\right)^2 \frac{d}{d\beta}\left(\frac{\sin [2k(\beta - \beta_1)b]}{2k(\beta - \beta_1)b}\right) ,$$

which is an odd function of β similar to the function defined by Equation (5). However, the peaks of the latter function are closer together (see Figure 2) which indicates that it might be superior to the former in resolving two plane waves whose angles of arrival are close together.

The requirement that we have access to the signal at every point in the aperture cannot be met; however, an approximation might be obtained by using a linear array of dipoles or horns instead of a continuous aperture. The horn outputs could be combined in pairs to approximate $\sigma(u) \cdot \delta(-u)$. The products could be summed sequentially with one receiver, or simultaneously with several receivers.

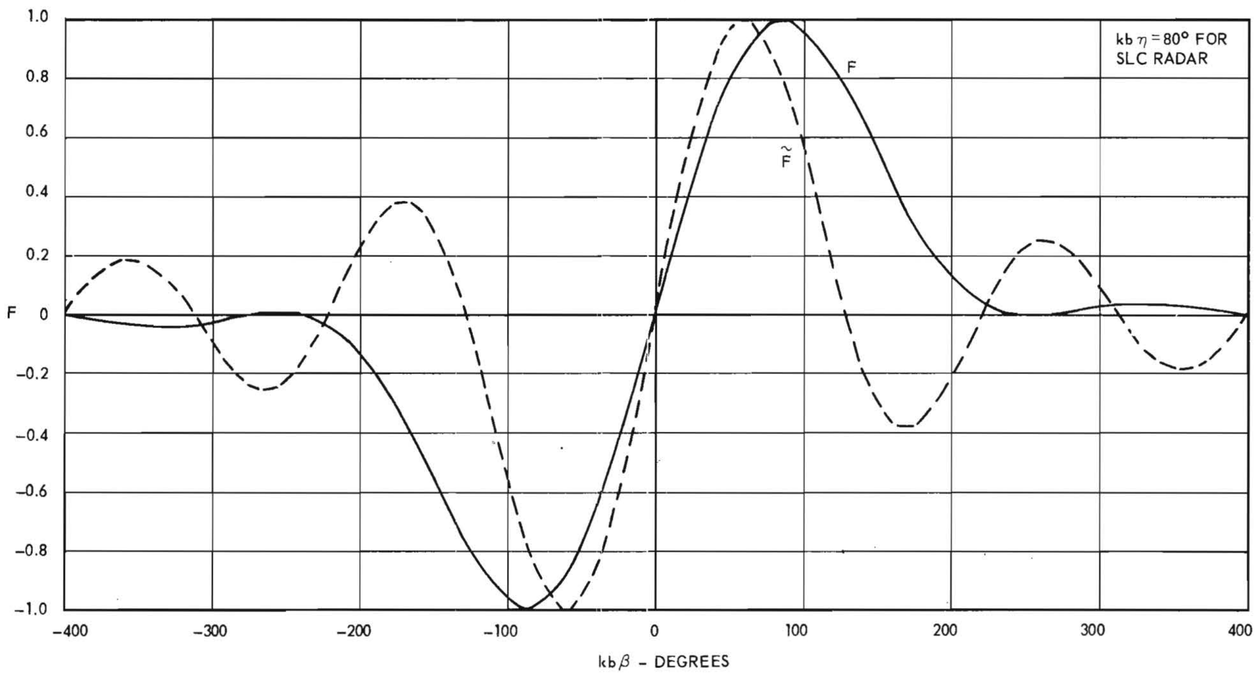

Figure 2. Normalized Error Signals for SLC Radar (F) and for Dot Product Scheme ($\tilde{F}$).

C. Convolution of Antenna Outputs.

Since these schemes would involve formidable engineering difficulties, an investigation was made to determine whether or not any operation in the image plane would produce $\tilde{F}(\beta,\beta_1)$. It was found that an equivalent function was defined by

$$F(\beta,\beta_1) = \int_{-\infty}^{\infty} \sum (y)\ \Delta(-y)\ dy \tag{10}$$

where $\sum (y)$ and $\Delta(y)$ represent respectively the even and odd functions integrated over the aperture, i.e., $\sum$ and Δ are signals such as might be obtained from a pair of feed horns in the image plane.

The study reported here was undertaken to investigate these ideas further, particularly the convolution scheme. The initial effort was devoted to determining the physical parameter corresponding to the variable y in Equation (10). It was found that y represents a phase shift which varies linearly over the aperture, and the use of swinging feeds would approximate the integration only if carried over very small limits. The question then arose as to whether an error signal obtained by integrating over finite limits would still exhibit superior resolution properties.

Two difficulties were encountered in trying to answer this question. First, the integral in (10) contains singularities and is not readily evaluated over finite limits. Second, since the phase shifts introduced by swinging feeds are not linear, a more exact expression for the error signal was needed.

The effort required to derive and evaluate such an expression seemed unwarranted without first having better evidence that the system might prove advantageous. Therefore a simplified version (the "four-horn case") was analyzed first.

II. THE FOUR HORN CASE

A. Geometry

In the four horn case we consider a linear aperture in which the field is sampled at only four points. This assumption simplifies the problem to one of adding up the signals from the points, rather than one of integrating over a continuous aperture. The geometry is illustrated in Figure 3 where AD represents an aperture through which the signal is admitted at only the four points A, B, C, and D. It is assumed that there exists a "focal point" at F at which a feed will receive in phase any signal arriving at A, B, C, and D in phase. We now wish to obtain an expression for the signal which would be received in the feed if it were swung in an arc about the aperture center from F to some other point P. This is done by summing up the signals arriving at P from the four points, after correcting each for the difference in path length traveled behind the aperture.

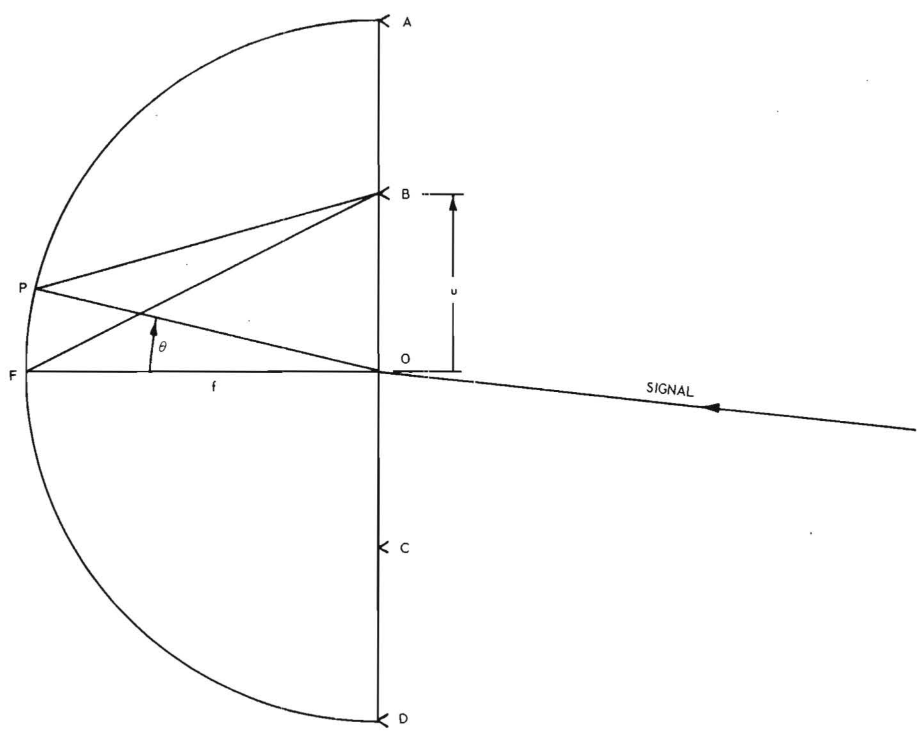

Figure 3. Four Horn Aperture Geometry.

Consider one of the four points, B, at a distance u from the center of the aperture. The difference in phase of a signal at F and P which is coming from B will be $k(\Delta y)$ where

(11) $$\Delta y = \overline{BF} - \overline{BP} = \sqrt{f^2 + u^2} - \sqrt{f^2 + u^2 - 2fu \sin \theta} \; .$$

To assign particular values to f and u, let

(12) $$f = b = \frac{1}{2} \text{ aperture width}$$

(13) $$u_A = u_D = b, \text{ and}$$

(14) $$u_B = u_C = \frac{b}{2} \; .$$

Combining Equations (2), (11), (12), (13) and (14) we obtain the four signals at P to be

(15)
$$S_{A+} = a \cos [\omega_1 t + kb(\beta + \eta - \beta_1) + kb\sqrt{2}\,(1 - \sqrt{1 - \sin \theta}) + \xi_1] \; ,$$

(16)
$$S_{B+} = a \cos [\omega_1 t + \frac{1}{2} kb(\beta + \eta - \beta_1) + \frac{kb}{2}(\sqrt{5} - \sqrt{5 - 4 \sin \theta}) + \xi_1] \; ,$$

(17)
$$S_{C+} = a \cos [\omega_1 t - \frac{1}{2} kb(\beta + \eta - \beta_1) + \frac{kb}{2}(\sqrt{5} - \sqrt{5 + 4 \sin \theta}) + \xi_1] \; ,$$

(18)
$$S_{D+} = a \cos [\omega_1 t - kb(\beta + \eta - \beta_1) + kb\sqrt{2}\,(1 - \sqrt{1 + \sin \theta}) + \xi_1 \; ,$$

where the subscript + indicates the signals are for the aperture whose axis is inclined at $+\eta$ to the tracking axis.

B. Derivation of Error Signals

Error signals were derived for three different systems and compared. Since only one signal was involved, ξ_1 was assumed to be zero.

1. Dot Product in the Aperture.

The error signal given by Equation (8) becomes

$$\tilde{F}(\beta,\beta_1) = \sigma_A\delta_D + \sigma_B\delta_C + \sigma_C\delta_B + \sigma_D\delta_A \tag{19}$$

where σ_i is the signal at the $i^{\underline{th}}$ point in the aperture, and δ_i is the signal after attenuation and phase shift. Since η is an unnecessary parameter in this case, it was set equal to zero. The σ's and δ's can thus be represented as

$$\sigma_A = a \cos\left[\omega_1 t + kb(\beta - \beta_1)\right] \quad , \tag{20}$$

$$\sigma_B = a \cos\left[\omega_1 t + \frac{1}{2} kb(\beta - \beta_1)\right] \quad , \tag{21}$$

$$\sigma_C = a \cos\left[\omega_1 t - \frac{1}{2} kb(\beta - \beta_1)\right] \quad , \tag{22}$$

$$\sigma_D = a \cos\left[\omega_1 t - kb(\beta - \beta_1)\right] \quad , \tag{23}$$

$$\delta_A = a \sin\left[\omega_1 t + kb(\beta - \beta_1)\right] \quad , \tag{24}$$

$$\delta_B = \frac{a}{2} \sin\left[\omega_1 t + \frac{1}{2} kb(\beta - \beta_1)\right] \quad , \tag{25}$$

$$\delta_C = -\frac{a}{2} \sin\left[\omega_1 t - \frac{1}{2} kb(\beta - \beta_1)\right] \quad , \tag{26}$$

$$\delta_D = -a \sin\left[\omega_1 t - kb(\beta - \beta_1)\right] \quad . \tag{27}$$

Substituting into (19), simplifying and dropping the $2\omega t$ terms (assumed to be removed by filtering) we have

$$\tilde{F}(\beta,\beta_1) = a^2\left[\sin 2kb(\beta - \beta_1) + \frac{1}{2} \sin kb(\beta - \beta_1)\right] \quad . \tag{28}$$

2. Ordinary Lobe Comparison

Let

$$E_1 = S_{A+} + S_{B+} + S_{C+} + S_{D+} \quad , \text{ and} \tag{29}$$

$$E_2 = S_{A-} + S_{B-} + S_{C-} + S_{D-} \quad , \tag{30}$$

where S_{K-} is the same as S_{K+} except that η is replaced by $-\eta$. Since we are assuming these signals are to be received at the focal point, θ is set equal to zero. The error signal will be

(31)
$$F(\beta,\beta_1) = E_1^{\,2} - E_2^{\,2} \quad .$$

Carrying out the required operations and dropping the $2\omega t$ terms

(32)
$$F(\beta,\beta_1) = a^2 \Big\{ 4 \sin \frac{kb(\beta - \beta_1)}{2} \sin \frac{k\beta\eta}{2} + 4 \sin \frac{3kb(\beta - \beta_1)}{2} \sin \frac{3kb\eta}{2} + 2 \sin 2\, kb(\beta - \beta_1) \sin 2kb\eta + 2 \sin kb(\beta - \beta_1) \sin kb\eta \Big\} \; .$$

3. Convolution Scheme.

In the convolution scheme, we assume that two feed pairs are being swung about the aperture center in the image plane. The two are swung in opposite directions; when one is displaced by an angle θ_1, the other is displaced by $-\theta_1$. One feed takes the sum of the signals from the two apertures, the other takes the difference, thus the feed outputs are

(33)
$$\Sigma\,(\theta) = E_1(\theta) + E_2(\theta) \quad , \text{ and}$$

(34)
$$\Delta\,(-\theta) = E_1(-\theta) - E_2(-\theta) \quad .$$

The error signal is obtained by multiplying the two outputs and integrating over all values of θ hence

(35)
$$F^{*}(\beta,\beta_1) = \int_{-\theta_1}^{\theta_1} \Sigma\,(\theta)\Delta(-\theta)\, d\theta \quad ,$$

where θ_1 is the outer limit of swing.

Substituting the values of S into Equation (35), simplifying and filtering out the $2\omega t$ terms gives

(36)

$$F^{*}(\beta,\beta_1) = 2a^2 \int_{-\theta_1}^{\theta} \Big\{ \sin 2kb\eta[\sin 2D - \sin 2kb(\beta - \beta_1)] + \sin kb\eta[\sin 2B - \sin kb(\beta - \beta_1)]$$

$$+ 2 \cos [\sqrt{2}\, kb - \frac{\sqrt{5}}{2} kb + A - C] \Big[\sin \frac{3kb\eta}{2} \cos \frac{kb(\beta - \beta_1)}{2} \sin (D + B)$$

$$- \sin \frac{3kb\eta}{2} \sin \frac{3kb(\beta - \beta_1)}{2} \cos (D - B) + \sin \frac{kb\eta}{2} \sin \frac{3kb(\beta - \beta_1)}{2} \sin (D - B)$$

$$- \sin \frac{kb\eta}{2} \sin \frac{kb(\beta - \beta_1)}{2} \cos (D + B) \Big] \Big\} d\theta \ ,$$

where

(37) $$A = \frac{kb}{2} \left[\sqrt{8} - \sqrt{5} + \sqrt{5 + 4 \sin \theta} - \sqrt{8(1 - \sin \theta)} \right] ,$$

(38) $$B = \frac{kb}{2} \left[\sqrt{8} - \sqrt{5} + \sqrt{5 - 4 \sin \theta} - \sqrt{8(1 - \sin \theta)} \right] ,$$

(39) $$C = \frac{kb}{2} \left[\sqrt{8} - \sqrt{5} + \sqrt{5 + 4 \sin \theta} - \sqrt{8(1 + \sin \theta)} \right] ,$$

(40) $$D = \frac{kb}{2} \left[\sqrt{8} - \sqrt{5} + \sqrt{5 - 4 \sin \theta} - \sqrt{8(1 + \sin \theta)} \right] .$$

C. Comparison of Error Curves

Using Equations (28), (32), and (36), error curves for the three systems were computed and compared. For the last two schemes, the value of η was chosen so that $kb\eta = \frac{\pi}{2}$. This choice gave some simplification of the equations, and is not very different from values of η which are used in SLC radars.

Other parameters were chosen to be $\lambda = 0.1$ ft. and $b = 3.0$ ft. These values result in a half-power beam width of approximately 1° for a continuous aperture.

The computations for F and $\tilde{F}$ were straight forward. For F^{*}, Equation (36) was programmed on the ERA 1101 digital computer, and the integration carried out numerically for $\theta_1 = 1^{\circ}$, 2°, and 3° (approximately 1, 2, and 3 beamwidths), and for $kb\beta$ ranging up to 360°. A comparison of the error functions is shown in Figure 4.

Using the distance between peaks of the error function as a measure of resolving ability, the convolution scheme does exhibit a small superiority over lobe comparison for two of the integration limits used. The peak of the curve for lobe comparison is located at $kb\beta = 75.1^{\circ}$. For the convolution scheme, the peaks are at 66.0°, 72.0°, and 76.5° for the integration limits of 1°, 2°, and 3° respectively. For comparison, the peak for the dot product scheme is at 49.7°.

After obtaining the above results, some additional calculations were carried out on the ERA 1101 to determine how the location of the peak varied for different integration limits. Since the location of error signal peaks also depends on aperture size, it was possible to take the lobe comparison scheme and compute the change in aperture size which would be required to give an error signal with the peak at any particular point. Such calculations were made for each value of integration limits used in Equation (36), thus giving the aperture change necessary for lobe comparison to match the convolution scheme in narrowness of the error signal peaks. The result was then considered to be the effective aperture increase of the convolution scheme, and is illustrated in Figure 5. The highest increase occured for θ_1 about 1.1° and was less than 14 percent. Such a value was considered too small to warrant any further investigation of this scheme.

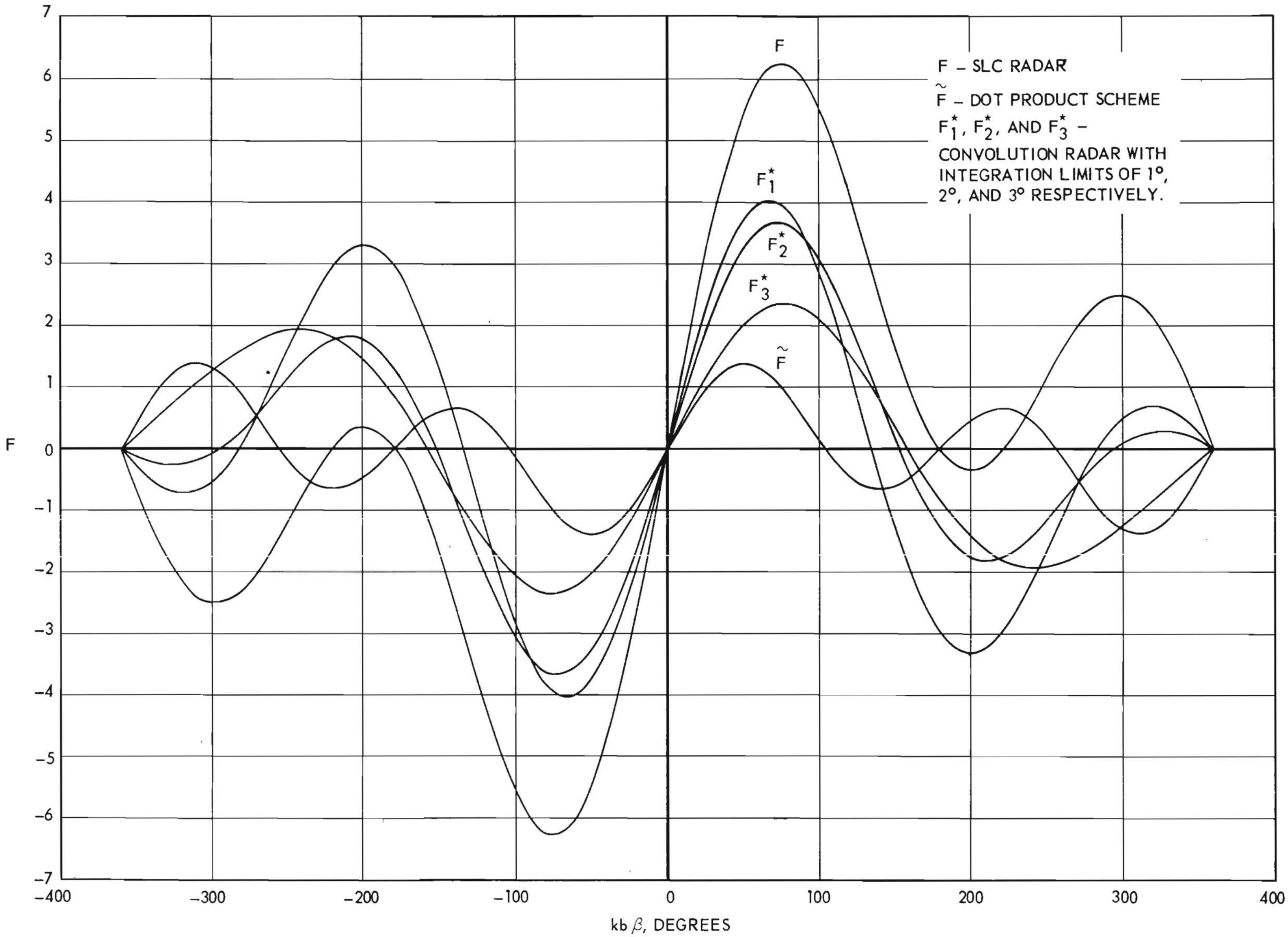

Figure 4. Comparison of Error Functions.

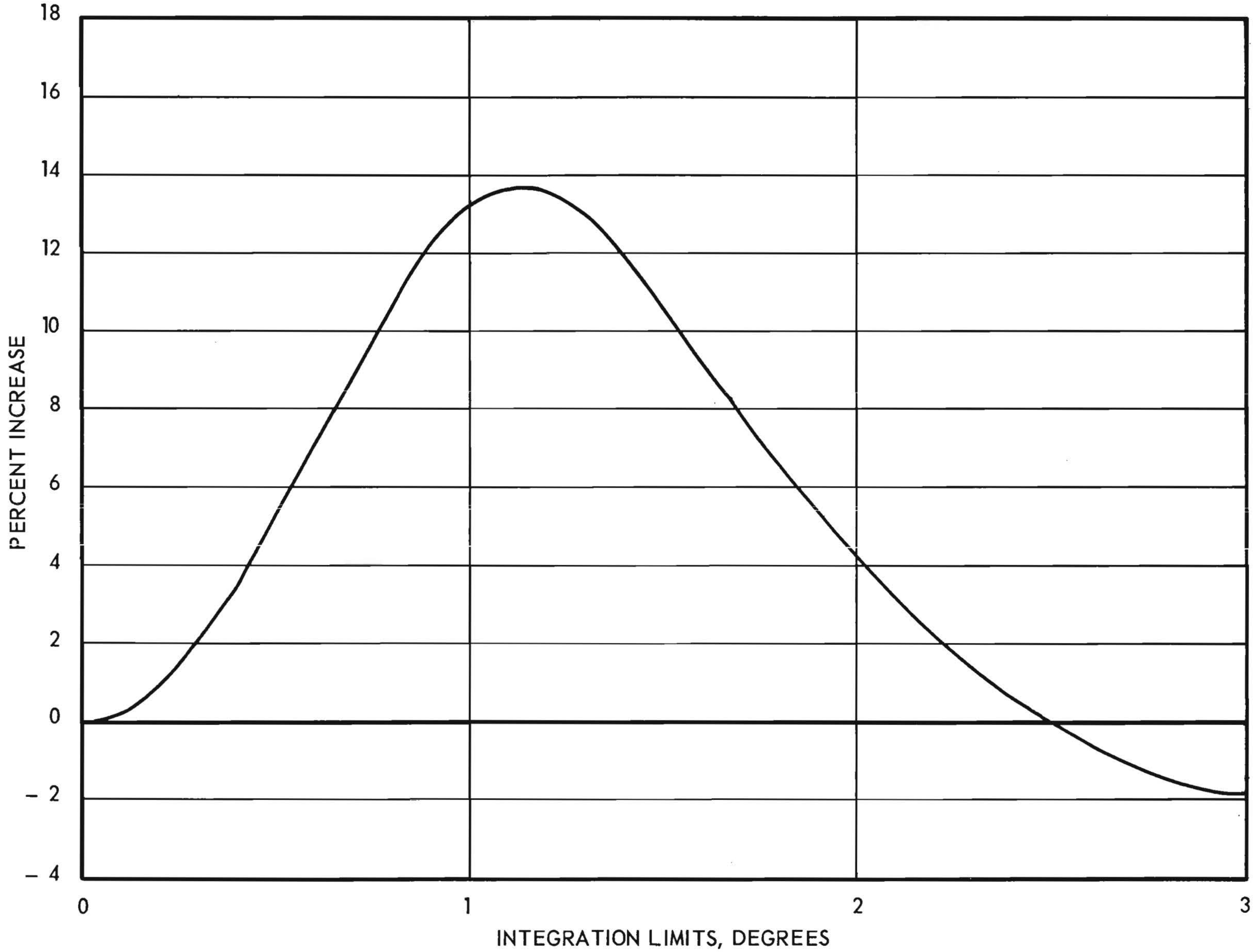

Figure 5. Effective Aperture Increase Achieved by Convolution.

III. ADDITIONAL STUDIES ON OPERATING IN THE APERTURE

A. Radar Applications

After the convolutions scheme was discarded, some additional study was made of the process of multiplying functions in the plane of the aperture. It was found that a function of $2kb\beta$ is obtained as an output when the signal is multiplied by an even (cosine) function of itself as well as when multiplied by an odd (sine) function.

This discovery led to a new viewpoint of the problem based on the type of multiplier used. In operational receivers, the incoming signal is multiplied by a locally generated signal (local oscillator) before detection and leads to an output which is a function of $kb\beta$. To compare the two, we derive the outputs of the two systems. First, when a locally generated signal is used we have

$$e = \cos \omega t \int_{-b}^{b} a \cos [\omega_1 t + k(\beta - \beta_1)u + \xi_1]\, du$$

(41) $$= ab \frac{\sin k(\beta - \beta_1)b}{k(\beta - \beta_1)b} \cos [(\omega - \omega_1)t - \xi_1] \quad .$$

Let

(42) $$f(x) = \frac{\sin kbx}{kbx} \quad ,$$

then

$$e = ab\ f(\beta - \beta_1) \cos [(\omega - \omega_1)t - \xi_1] \quad .$$

After detection, the final output is of the form

(43) $$E = ab\ f(\beta - \beta_1) \quad .$$

Now consider the case of using a function of the signal as a multiplier. The treatment will be restricted to using the cosine function, since its (even) output is directly comparable. We have

$$\tilde{E} = (\tfrac{a}{2})^2 \int_{-b}^{b} \cos [\omega_1 t + k(\beta - \beta_1)u + \xi_1] \cos [\omega_1 t - k(\beta - \beta_1)u + \xi_1]\, du$$

$$\tilde{E} = b\, (\tfrac{a}{2})^2 f(2\beta - 2\beta_1). \tag{44}$$

A plot of E and $\tilde{E}$ against β is shown in Figure 6 (the amplitudes are arbitrary). It appears that if the multiplier is derived from the signal itself instead of from an independent source, the system will have greater resolving power. To investigate this point, let us assume that two signals of equal amplitude are arriving at angles β_1 and β_2. When a local oscillator is used, we have

$$e = \cos \omega t \int_{-b}^{b} \left\{ a \cos [\omega_1 t + k(\beta-\beta_1)u + \xi_1] + a \cos [\omega_2 t + k(\beta-\beta_2)u + \xi_2] \right\} du$$

$$= ab \left\{ f(\beta-\beta_1) \cos [(\omega_1-\omega)t + \xi_1] + f(\beta-\beta_2) \cos [(\omega_2-\omega)t + \xi_2] \right\} . \tag{45}$$

After detection, the output is

$$E = ab \sqrt{f^2(\beta-\beta_1) + f^2(\beta-\beta_2) + 2f(\beta-\beta_1)f(\beta-\beta_2) \cos [(\omega_2-\omega_1)t + \xi_2 - \xi_1]} . \tag{46}$$

When the even function of the signal is used as a multiplier, we have

$$E = (\tfrac{a}{2})^2 \int_{-b}^{b} \left\{ \cos [\omega_1 t + k(\beta-\beta_1)u + \xi_1] + \cos [\omega_2 t + k(\beta-\beta_2)u + \xi_2 \right\} \left\{ \cos [\omega_1 t - k(\beta-\beta_1)u + \xi_1] + \cos [\omega_2 t - k(\beta-\beta_2)u + \xi_2] \right\} du.$$

$$(47) \qquad \tilde{E} = b\left(\frac{a}{2}\right)^2 \left\{ f(2\beta-2\beta_1) + f(2\beta - 2\beta_2) + 2\cos\left[(\omega_2-\omega_1)t + \xi_2-\xi_1\right] f(\beta_2+\beta_1-2\beta) \right\} .$$

Comparing Equations (46) and (47), the two outputs are functionally similar. Each consists of a dc term due to each of the two signals plus a cross product term which is time modulated at a frequency $(\omega_2 - \omega_1)t$. For radar applications, this modulation frequency might be anything from zero up to thousands of cycles per second, depending on the relative motion of the two targets. For targets maintaining formation, however, the frequency would be only a few cycles per second at most, and often would be only a fraction of a cycle per second. It would not be practical to remove such a frequency by filtering, hence the output would be a function of time. In effect, the two signals can be regarded as "coherent" and the output will fluctuate between the "in-phase" and "out-of-phase" condition.

A series of graphs of E and $\tilde{E}$ for in-phase and out-of-phase conditions were prepared, a sample of which is shown in Figure 7. In general they show that for $k(\beta_2 - \beta_1)\, b \leq 240^\circ$, both E and $\tilde{E}$ exhibits a single peak for the in-phase condition. For the out-of-phase condition, the graph of E has two humps, and the graph of $\tilde{E}$ has three. If we take as the point of resolution that separation of β_1 and β_2 which will separate the in-phase output into a double peak, then resolution occurs first for the system with a local oscillator, when $k(\beta_2 - \beta_1)b = 240^\circ$. This case is shown in Figure 8, and it can be seen that the other system has not yet resolved the two. It therefore appears that a radar system which uses a function of its signal as a multiplier will be inferior to operational type radars in angular resolution of two interfering waves.

B. Direction Finding

Although the dot-product in the aperture scheme does not appear suitable for radar applications, it is possible that some other microwave equipment might use it advantageously. For instance, consider a direction finding device which consists essentially of only a receiver, with the transmitters

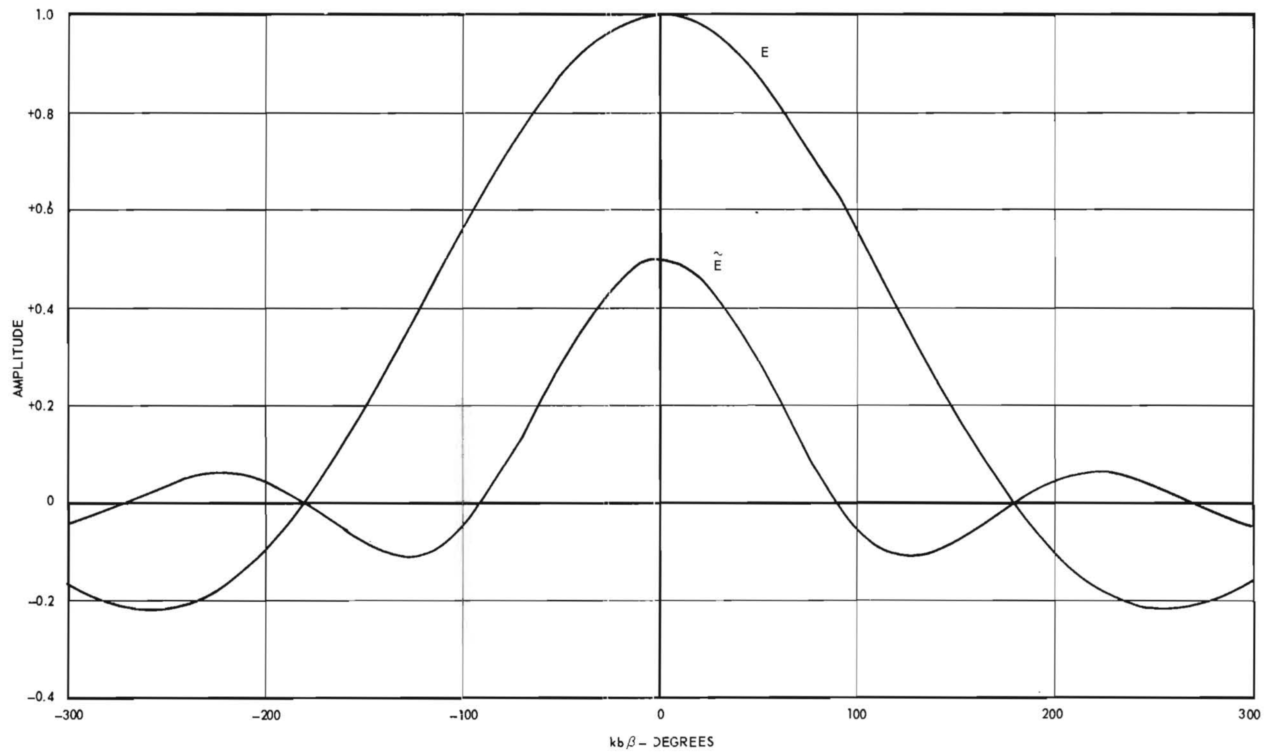

Figure 6. E and Ẽ for One Incident Wave.

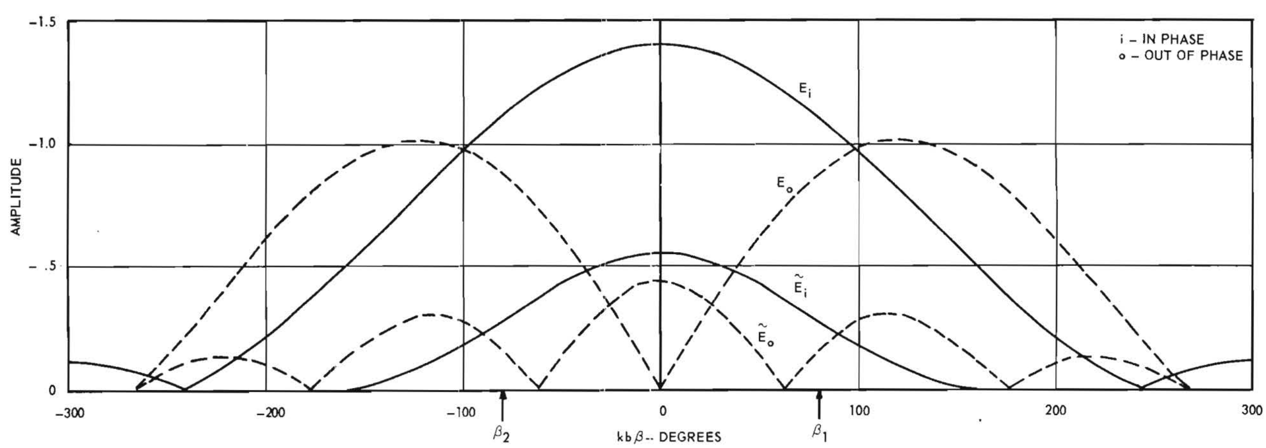

Figure 7. E and Ẽ for 160° Separation of Two Coherent Signals.

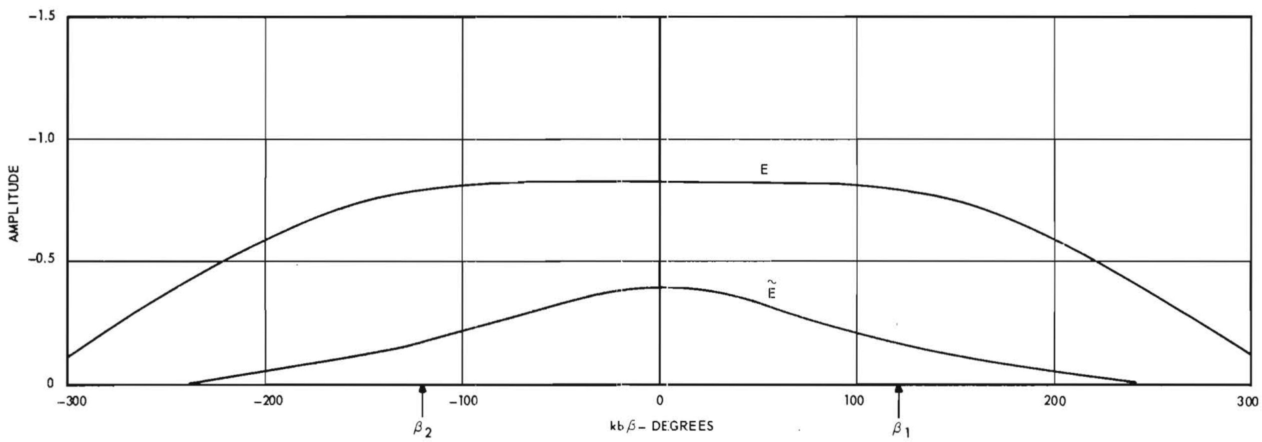

Figure 8. E and Ẽ for 240° Separation of Two Coherent Signals Arriving in Phase.

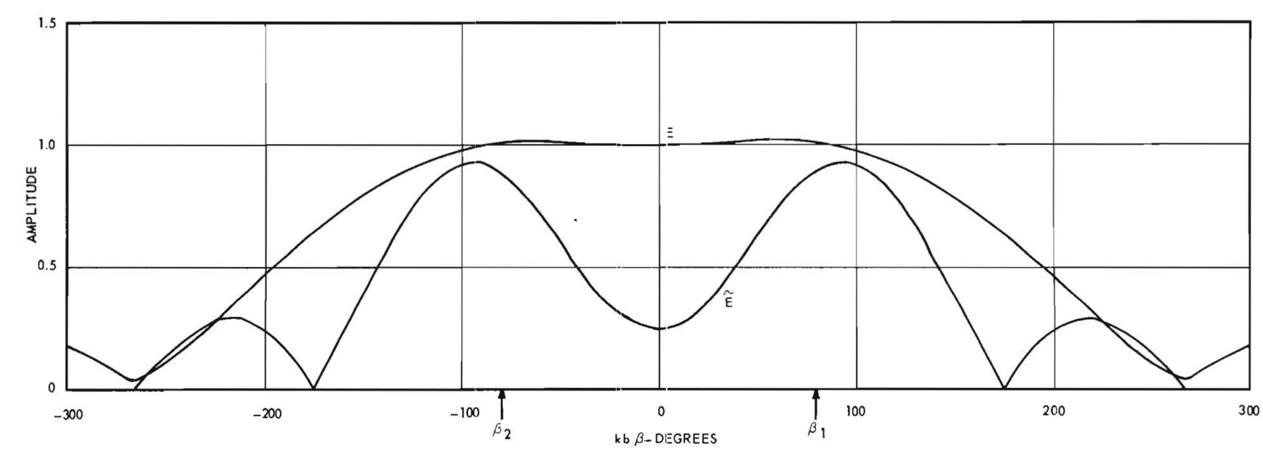

Figure 9. E and Ẽ for 160° Separation of Two Incoherent Signals.

located on the targets. Such a device might be used for traffic control of friendly aircraft. By selective tuning of the transmitters, the frequencies might be made sufficiently different so that the cross product terms in Equations (46) and (47) could be filtered out. Thus we would have only the two steady state terms corresponding to the "non-coherent" case. The dot-product of signals will then give better resolution as illustrated in Figure 9.

Not much effort has been devoted to this type of application and it is mentioned here simply for completeness.

IV. CONCLUSIONS

In summary, the following conclusions were reached:

1. It is not possible to approximate the dot-product-in-the-aperture scheme by swinging feeds in the image plane of an antenna.
2. For radar applications, the dot-product scheme is not superior to ordinary radars in angular resolution of two interfering waves.
3. The dot-product scheme might still be useful in some application other than radar.

The author wishes to extend his thanks to Mr. W. W. Wright for his suggestions and aid in carrying out this study.

Respectfully submitted:

R. D. Wetherington
Project Director

Approved:

J. E. Boyd, Associate Director
Engineering Experiment Station